RPERESENTATION THEORY MIDTERM EXAMINATION

You can quote any theorem done in class (unless you are being asked to prove it), please write down a complete and clear statement of the result you are quoting.

(1) Define the group algebra of any finite group G over a field k. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein 4 group. Prove that the group algebra $k[G]$ is isomorphic to the k algebra $k[x,y]/(x^2, y^2)$. (6 marks)

(2) Let p be a prime, let k be an algebraically closed field of characteristic p, and let $G = \mathbb{Z}_p$ be the cyclic group of order p. Determine all finite dimensional irreducible representations of G over k. (6 marks)

(3) Let $G = \mathbb{Z}_3$ be the cyclic group of order 3, let g denote a generator of G. Let $\theta : G \to GL_2(\mathbb{R})$ be a two dimensional representation of G over $\mathbb{R}$ which sends g to the 2×2 real matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Check that θ defines a representation. Is θ an irreducible representation? (6 marks)

(4) Let G be a finite group, let $g \in G$ be a central element of G. Let $\rho : G \to GL_2(\mathbb{Q})$ be a two dimensional representation of G over $\mathbb{Q}$ such that $\rho(g)$ is the 2×2 rational matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Prove that ρ is a reducible representation. (6 marks)

(5) Write down explicitly all finite dimensional irreducible representations (up to isomorphism) of S_3 over $\mathbb{C}$. Justify why these are irreducible and also (by quoting relevant reseults) why your list is complete. (8 marks)

(6) Let k be any field, let $M_n(k)$ be the algebra of all $n \times n$ matrices over k (for some positive integer n). Prove that $M_n(k)$ is a semisimple k-algebra, and find all simple $M_n(k)$ modules (upto isomorphism). (8 marks)